

TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
HOMEWORK 0

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Homework 0.1 (Warmup). Give an example of the following objects or show they cannot exist, respectively.

- a. A holomorphic function $f: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ such that $f'(z) = 1/z$.
- b. A holomorphic function $f: \mathbb{C} \setminus \{\Re \leq 0\} \rightarrow \mathbb{C}$ with $f'(z) = 1/z$? Here $\{\Re \leq 0\}$ designates the set of all complex numbers with nonpositive real part.
- c. A domain $D \subset \mathbb{C}$ and a nonconstant holomorphic function $f: D \rightarrow \mathbb{C}$ such that $|f|$ is constant.
- d. A nonconstant holomorphic function $f: \mathbb{C} \rightarrow \mathbb{C}$ such that $f(n) = 0$ for every $n \in \mathbb{N}$.
- e. A nonconstant holomorphic function $f: \mathbb{C} \rightarrow \mathbb{C}$ such that $f(i + 1/n) = 0$ for every $n \in \mathbb{N}$.
- f. A nonconstant holomorphic function $f: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ that is bounded.

Homework 0.2 (Image of entire functions). Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a non-constant entire function (i.e. f is holomorphic on the entire complex plane). Show $f(\mathbb{C})$ is dense in $\mathbb{C}$ ¹.

Homework 0.3 (Complex Arzelà–Ascoli theorem). Let $K \subset \mathbb{C}$ be a compact set and let $(f_n)_{n \in \mathbb{N}}$ be a *uniformly bounded* sequence of functions $f_n: K \rightarrow \mathbb{C}$, i.e.

$$\sup_{n \in \mathbb{N}} \sup_{z \in K} |f_n(z)| < \infty.$$

- a. Let $S \subset K$ be a countable set. Show there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that the point evaluations $(f_{n_k}(z))_{k \in \mathbb{N}}$ converge to some value $f_z \in \mathbb{C}$ for every $z \in S$.
- b. Now assume in addition that $(f_n)_{n \in \mathbb{N}}$ is *equicontinuous*, i.e. for every $\varepsilon > 0$ and every $z \in K$ there exists $\delta > 0$ such that for every $x \in K$,

$$|x - z| \leq \delta \implies \sup_{n \in \mathbb{N}} |f_n(x) - f_n(z)| \leq \varepsilon.$$

Suppose moreover the set S from a. is dense in K . Show that the point evaluations $(f_{n_k}(z))_{k \in \mathbb{N}}$ converge to some value $f_z \in \mathbb{C}$ for every $z \in K$.

- c. Retain the setting of b. and define the function $f: K \rightarrow \mathbb{C}$ by $f(z) := f_z$. Show $(f_{n_k})_{k \in \mathbb{N}}$ converges uniformly to f on K . Conclude f is continuous.

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¹**Hint.** Argue by contradiction and use Liouville’s theorem for a suitable function.